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NON-INTERVAL GREEDOIDS AND THE TRANSPOSITION PROPERTY*

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In previous papers we have mainly studied greedoids with the interval property. This paper exhibits 11 classes of greedoids whose members do not necessarily have the interval property. These non-interval greedoids are related to some fundamental algorithms and procedural principles like Gaussian elimination, blossom trees, series–parallel decomposition, ear decomposition, retracting and dismantling. We introduce some weaker exchange properties. One of them can be shown to be equivalent to the greedoid exchange property. Another one leads to the definition of transposition greedoids. Besides all interval greedoids, some non-interval greedoids share the transposition property.

1. Introduction and basic definitions

In previous papers we have introduced greedoids as extensions of matroids and we have mainly studied those greedoids which have the 'interval' property. Most of the structural and algorithmic results obtained so far deal with interval greedoids. While studying these interval greedoids we found some interesting classes of greedoids whose members do not necessarily have the interval property. We call those for short *non-interval greedoids*. These non-interval greedoids are related to some fundamental algorithmic and procedural principles like Gaussian elimination, blossom trees, series–parallel decomposition, ear decomposition, retracting and dismantling. Hence, a further study of them might give some additional insight into algorithmic aspects of greedoids.

In this paper we study the common combinatorial structure of various recursive deletion procedures. Some of these (like shelling away endpoints of a tree, or vertices of the convex hull of a pointset in $\mathbb{R}^n$, or minimal elements of a poset) lead to greedoids with the interval property, and were studied in previous papers

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(Korte and Lovász [13], Crapo [2]). Others, (like series–parallel decomposition, retracting and dismantling) still yields greedoids. These are not interval greedoids, but have instead a slightly weaker property called the transposition property.

In this section we will give some basic definitions and facts about greedoids which are needed for the following. Section 2 gives a list of 11 classes of non-interval greedoids. Some of them were already introduced in Korte and Lovász [12]. In Section 3 we introduce a weak exchange property for set systems. It is an interesting fact that this weak exchange property is equivalent to the stronger exchange property used in previous papers and hence is sufficient for the definition of greedoids. Section 4 is the main part of this paper. There we introduce a certain exchange property which we call the transposition property. We prove that this property already implies the greedoid exchange property. It is an easy observation that every interval greedoid has the transposition property. Moreover, we show that some of the non-interval greedoids introduced in Section 2 also enjoy this property. This fact also leads to a simpler proof of the result that these structures are indeed greedoids. We feel that further investigations along this line might lead to a better understanding of non-interval greedoids. Finally, in Section 5 we explain by appropriate counterexamples that some non-interval greedoids do not satisfy the transposition property.

We give now some definitions and basic facts about greedoids. For more details the reader is referred to Korte and Lovász [9, 10, 12–14].

A *set system* over a finite ground set E is a pair $(E, \mathcal{F})$ with $\mathcal{F} \subseteq 2^E$. A set system is a *matroid* if the following axioms hold.

(M1) $\emptyset \in \mathcal{F}$.

(M2) $X \subseteq Y \in \mathcal{F}$ implies $X \in \mathcal{F}$.

(M3) if $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there exists an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.

A set system which satisfies only (M1) and (M2) is called an *independence system* or hereditary set system. For an arbitrary set system we define its *hereditary closure* as

$$\mathcal{H}(\mathcal{F}) := \mathcal{H} = \{X \subseteq Y \colon Y \in \mathcal{F}\}$$

and its *accessible kernel* as

$$\mathcal{K}(\mathcal{F}) := \mathcal{K} = \{X \in \mathcal{F} \colon X = \{x_1, \ldots, x_k\} \text{ and } \{x_1, \ldots, x_i\} \in \mathcal{F} \text{ for all } 1 \leq i \leq k\}.$$

Sets belonging to $\mathcal{F}$ are called *feasible sets* (or in case of a hereditary set system *independent sets*). Elements of $2^E - \mathcal{F}$ are *non-feasible sets* (or *dependent sets*). For $X \subseteq E$ a maximal feasible subset of X is called a *basis* of X. A set which contains a basis is called *spanning*.

Greedoids were introduced in Korte and Lovász [7]. They are generalizations (or in a sense ordered versions) of matroids. A *language* $\mathscr{L}$ over a finite ground set E (which is called the *alphabet*) is a collection of *finite sequences* $x_1 \cdots x_k$ of elements $x_i \in E$ for $1 \leqslant i \leqslant k$. We call these sequences *strings* or *words*. Words will be denoted by small greek letters. E^* denotes the set of all words over the alphabet E. Thus $\mathscr{L} \subseteq E^*$. By deleting certain letters of a word but keeping the order of the remaining we get a *subword*. The underlying set of a word α is denoted by $\tilde{\alpha}$, $\tilde{\mathscr{L}} \subseteq 2^E$ is the collection of all underlying sets of $\mathscr{L}$. Similarly to the cardinality symbol we use $|\alpha|$ to denote the length of a string α. The notation $x \in \alpha$ means $x \in \tilde{\alpha}$. The concatenation of two words α, β is denoted by $\alpha \cdot \beta$, i.e., the string α followed by the string β. A language is called *simple* if no letter is repeated in any of its words, i.e., $|\alpha| = |\tilde{\alpha}|$ for every $\alpha \in \mathscr{L}$.

$(E, \mathscr{L})$ is called a *hereditary language* if

(G1) $\emptyset \in \mathscr{L}$.

(G2) if $\alpha \in \mathscr{L}$ and $\alpha = \beta \cdot \gamma$, then $\beta \in \mathscr{L}$, i.e., every beginning section of a word belongs to the language.

A simple hereditary language is called a *greedoid* if in addition the following holds:

(G3) if $\alpha, \beta \in \mathscr{L}$ and $|\alpha| > |\beta|$, then there exists an $x \in \alpha$ such that $\beta \cdot x \in \mathscr{L}$.

Again, a word is called *feasible* if it belongs to $\mathscr{L}$. Maximal feasible words are called *basic words*. A greedoid is called full if its basic words contain all letters from E.

Apart from this definition of hereditary languages and greedoids as collections of *ordered* sets, we can also define them in an unordered version by considering the underlying sets of strings. Then an *accessible set system* $(E, \mathscr{F})$ is a set system $\mathscr{F} \subseteq 2^E$ with

(H1) $\emptyset \in \mathscr{F}$.

(H2) for all $X \in \mathscr{F} - \{\emptyset\}$ there exists an $x \in X$ such that $X - x \in \mathscr{F}$.

If $(E, \mathscr{F})$ is a (not necessarily accessible) set-system, then it contains a unique largest subsystem $(E, \bar{\mathscr{F}})$ which is accessible. We call $\bar{\mathscr{F}}$ the *accessible kernel* of $(E, \mathscr{F})$. A set system $(E, \mathscr{F})$ is a *greedoid* if (H1), (H2) and the following hold:

(H3) if $X, Y \in \mathscr{F}$ and $|X| = |Y| + 1$, then there exists an $x \in X - Y$ such that $Y \cup x \in \mathscr{F}$.

Given a simple hereditary language $(E, \mathscr{L})$ satisfying (G1) and (G2), we call the set system

$$\mathscr{F} = \{\{x_1, \ldots, x_k\} : x_1 \cdots x_k \in \mathscr{L}\}$$

induced by $\mathscr{L}$. It is immediate that $(E, \mathscr{F})$ is an accessible set system, i.e., it satisfies (H1) and (H2). Starting from a given accessible set systems $(E, \mathscr{F})$, we can introduce an inverse construction

$$\mathscr{L} = \{x_1 \cdots x_k \colon \{x_1, \ldots, x_i\} \in \mathscr{F} \text{ for } 1 \leq i \leq k\}.$$

We call $(E, \mathscr{L})$ the hereditary language *induced by* the accessible system $(E, \mathscr{F})$. However, these operations are not inverse for general accessible set systems or hereditary languages. Let $(E, \mathscr{L})$ be a simple hereditary language and $(E, \mathscr{F})$ the accessible set system induced by the above construction. Take $(E, \mathscr{F})$ and construct

$$\mathscr{L}' = \{x_1 \cdots x_k \colon \{x_1, \ldots, x_i\} \in \mathscr{F} \text{ for } 1 \leq i \leq k\}$$

as above. Then $\mathscr{L} \subseteq \mathscr{L}'$. But equality holds iff a special case of the greedoid exchange property (G3) holds. This will be shown in Section 4. This implies that for a greedoid the definition with (G1), (G2) and (G3) is indeed equivalent to that with (H1), (H2) and (H3), as it was already shown in Korte and Lovász [12]. In the following we will use both definitions concurrently. It is an easy observation that (H2) and (H3) are equivalent to (M3). Hence, $(E, \mathscr{F})$ is a greedoid iff (M1) and (M3) hold.

Therefore we can consider greedoids either as simple hereditary languages satisfying ordered versions (G1), (G2) and (G3) of the matroid axiom or as unordered set systems satisfying (M1) and (M3), i.e., direct relaxations of matroids. For short we call them the ordered or the unordered version of greedoids.

A fundamental property of certain greedoids is the *interval property*. We say that a greedoid $(E, \mathscr{F})$ has the interval property if for all $A \subseteq B \subseteq C \subseteq E$ and $x \in E - C$ with $A, B, C \in \mathscr{F}$ such that $A \cup x, C \cup x \in \mathscr{F}$ it follows that $B \cup x \in \mathscr{F}$. A greedoid with the interval property is called for short an *interval greedoid.* Björner [1] has shown that a simple hereditary language $(E, \mathscr{L})$ is an interval greedoid iff instead of (G3) we have

(G3′) If $\alpha, \beta \in \mathscr{L}$ with $|\alpha| > |\beta|$, then there exists a subword α' of α with $|\alpha'| \geq |\alpha| - |\beta|$ such that $\beta \cdot \alpha' \in \mathscr{L}$.

2. Some examples of non-interval greedoids

In the following we give a list of 11 classes of non-interval greedoids, some of which were mentioned in previous papers.

2.1. *Bipartite matching greedoids*

Let G be a bipartite graph with bipartition $V(G) = U \cup W$. Let $U = \{u_1, \ldots, u_n\}$, and define

$$\mathscr{F} = \{X \subseteq W \colon X \text{ can be matched in } G \text{ with } \{u_1, \ldots, u_{|X|}\}\}.$$

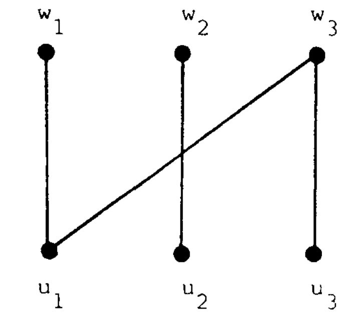

Fig. 1.

Then $(E, \mathscr{F})$ is a greedoid. Property (M1) is obvious. To prove (M3), let $X, Y \in \mathscr{F}$, $|X| > |Y|$. Then by the definition of $\mathscr{F}$, there exist matchings $M_1 \subseteq E(G)$ and $M_2 \subseteq E(G)$ such that M_1 matches X with $\{u_1, \ldots, u_{|X|}\}$ and M_2 matches Y with $\{u_1, \ldots, u_{|Y|}\}$. Consider $M_1 \cup M_2$. Since M_2 does not cover $u_{|Y|+1}$, but M_1 does, there is an alternating $M_1 - M_2$-path P starting from $u_{|Y|+1}$. Since this path P always enters U on an M_2-edge, and therefore it can always continue on an M_1-edge, it must end with an M_1-edge at a point $x \in X$. So $M_2' = (M_2 - P) \cup (P - M_2)$ is a matching which matches $Y \cup \{x\}$ with $\{u_1, \ldots, u_{|Y|+1}\}$. Hence $Y \cup \{x\} \in \mathscr{F}$.

The bipartite graph in Fig. 1 shows that bipartite matching greedoids do not necessarily have the interval property. In fact, $\emptyset, \{w_1\}, \{w_1, w_2\}, \{w_3\}, \{w_1, w_2, w_3\} \in \mathscr{F}$ but $\{w_1, w_3\} \notin \mathscr{F}$.

2.2. *Gaussian elimination greedoids*

This class of greedoids was observed by Goecke [5]. Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ be any matrix. Let $E = \{1, \ldots, n\}$ and let $\mathscr{F}$ consist of those subsets $\{j_1, \ldots, j_k\} \subseteq E$ for which the submatrix $(a_{ij_v})_{i,v=1}^{k}$ is non-singular. Then $(E, \mathscr{F})$ is a greedoid. The name for these greedoids comes from the observation that if we do Gaussian elimination row by row, then the column indices of pivot elements will form a feasible word in this greedoid. It is not difficult to see that bipartite matching greedoids are special Gaussian elimination greedoids. This follows by the same argument as the linear representability of transversal matroids (see Edmonds [4]). Hence they are not interval greedoids either.

2.3. *Perfect elimination greedoids*

An edge xx' of a bipartite graph is called *bisimplicial* if the set of vertices adjacent to x and x' induces a complete bipartite graph with bipartition $V(G) = U \cup W$. A sequence $(u_1w_1, u_2w_2, \ldots, u_kw_k)$ of edges of G is called an *elimination sequence* if u_iw_i is a bisimplicial edge in $G - u_1 - w_1 - \cdots - u_{i-1} - w_{i-1}$. Then let $E = U$ and let $\mathscr{L} = \{u_1 \cdots u_k\colon u_i \in U$ and there exist $w_1, \ldots, w_k \in W$ such that $(u_1w_1, \ldots, u_kw_k)$ is an elimination sequence$\}$. It was proved in Korte and Lovász [8] by a rather lengthy argument that $(E, \mathscr{L})$ is a greedoid. The

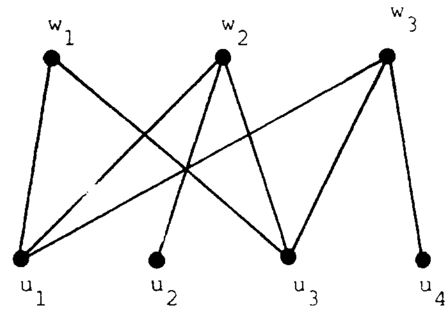

Fig. 2.

bipartite graph in Fig. 2 defines a perfect elimination greedoid which does not have the interval property. In fact, (u_1w_1, u_2w_2, u_3w_3) is an elimination sequence and hence $u_1u_2u_3 \in \mathscr{L}$. Furthermore, u_3w_1 is bisimplicial, and so $u_3 \in \mathscr{L}$. But u_1 is incident with only one bisimplicial edge u_1w_1, and in $G - u_1 - w_1$, u_3 is not incident with any bisimplicial edge. Thus $u_1u_3 \notin \mathscr{L}$. Perfect elimination greedoids are connected to perfect Gaussian elimination (see Golumbic [6]).

2.4. *Series–parallel reduction greedoids*

Let G be a loopless graph. An edge of G is called *reducible* if it is either a pending edge, or is parallel with another edge, or is in series with another edge (i.e., they have a common endpoint of degree 2). If e is reducible in G, we define $G \div e$ as the graph obtained from G by deleting e if e has a parallel edge and by contracting e if e is a pending edge or is in series with another edge. If e is both parallel and in series, then we delete it. So we do not produce any loops.

A sequence $e_1 \cdots e_k$ of edges is called a *series–parallel reduction sequence* if e_i is reducible in $G \div e_1 \div \cdots \div e_{i-1}$ for $i = 1, \ldots, k$. Let $E = E(G)$ and let $\mathscr{L}$ denote the set of all series–parallel reduction sequences. Then $(E, \mathscr{L})$ is a greedoid, as it will be shown later. It does not necessarily have the interval property, as shown by the graph in Fig. 3, where there $acb \in \mathscr{L}$ and $b \in \mathscr{L}$ but $ab \notin \mathscr{L}$. If G is a tree, then the resulting greedoid is an edge-shelling of the tree.

One could slightly modify the definition of a reducible edge, by replacing "pending edge" by "coloop" and "two edges in series" by "two edges forming a cut". Then the definition of series parallel reduction greedoids could be extended to matroids.

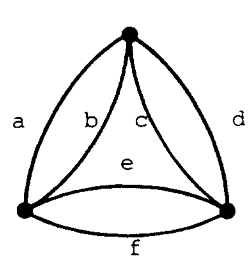

Fig. 3.

2.5. *Retract greedoids*

Let G be a digraph, and $x, y \in V(G)$. We say that x is *retractable to* y in G if for any edge $xz \in E(G)$ $(z \neq x)$ we also have $yz \in E(G)$ and for any edge $zx \in E(G)$ $(z \neq x)$ we have $zy \in E(G)$, and if $xx \in E(G)$, then $yy \in E(G)$. In other words, the mapping $\rho_{xy} : V(G) \to V(G) - x$ defined by

$$\rho_{xy}(z) = \begin{cases} y & \text{if } z = x, \\ z & \text{if } z \neq x \end{cases}$$

is edge-preserving. We say that x is *retractable* in G if it is retractable to some point $y \neq x$. A sequence $x_1 \cdots x_k$ of points is called *retract sequence* if x_i is retractable in $G - x_1 - \cdots - x_{i-1}$ for $i = 1, \ldots, k$. In Korte and Lovász [8] we showed that retract sequences form a greedoid on the ground set $E = V(G)$. Again, this proof was rather complicated. Within the framework of transposition greedoids we shall obtain a shorter argument for this. This construction generalizes the retracts of partial orders, for which closely related results were obtained by Duffus and Rival [3].

The retract greedoid of the graph in Fig. 4 (which in fact corresponds to a poset) does not have the interval property.

This construction can be even further generalized. Let S be a transformation monoid on a finite set E (i.e., let S be a set of mappings of E into itself, containing id_E and closed under composition). A subset $X \subseteq E$ is called a *retract of* E if there is an idempotent element $\varphi \in S$ such that $X = \varphi(E)$. A sequence $x_1 \cdots x_k$ of distinct elements of E is called a *retract sequence* if $E - x_1 - \cdots - x_i$ is a retract of E for $i = 1, \ldots, k$. It will follow from the results of Section 4 that these retract sequences form a greedoid.

Note that this greedoid can even easier be defined in the unordered version: its feasible sets are the members of the accessible kernel of the set-system

$$\{E - \varphi(E) \colon \varphi \in S,\ \varphi^2 = \varphi\}.$$

2.6. *Dismantling greedoids*

Let G be a digraph and $x \in V(G)$. We say that x is *dismantlable* if there exists a $y \neq x$ such that x is retractable to y and x and y are adjacent in G. A sequence

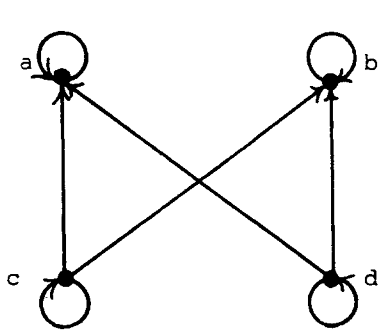

Fig. 4.

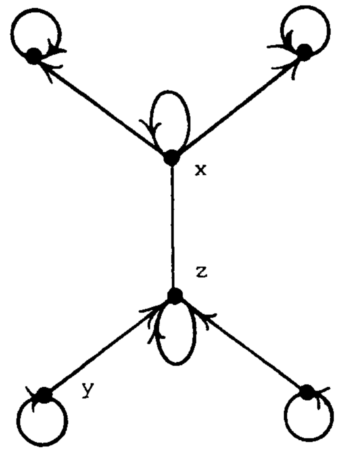

Fig. 5.

$x_1 \cdots x_k$ is called a *dismantling sequence* if x_i is dismantlable in $G - x_1 - \cdots - x_{i-1}$. Then dismantling sequences form a greedoid on $V(G)$. This again generalizes a construction of Duffus and Rival [3] for partially ordered sets. If P is a partial order and $x \in P$, then x is called *dismantlable* if it has a unique upper cover or a unique lower cover. For the relation graph of the partial order, this coincides with the above definition. Dismantling greedoids are not necessarily interval greedoids even in the special case of dismantling partially ordered sets, as shown by the partial order in Fig. 5. There xyz and z are dismantling sequence but xz is not.

2.7. *Twisted matroids*

This construction is due to Björner [1]. Let $(E, \mathcal{M})$ be a matroid and $A \in \mathcal{M}$. Define

$$\mathcal{F} = \{X \triangle A : X \in \mathcal{M}\}$$

(where $\triangle$ denotes symmetric difference). Then $(E, \mathcal{F})$ is a greedoid, but not necessarily an interval greedoid: If $(E, \mathcal{M})$ is the 2-uniform matroid on $E = \{a, b, c\}$ and $A = \{b\}$, then $\{a\} \in \mathcal{F}$, $\{a, b\} \in \mathcal{F}$, $\{c\} \in \mathcal{F}$, $\{a, b, c\} \in \mathcal{F}$ but $\{a, c\} \notin \mathcal{F}$.

Finally we list some more classes of greedoids which are not in general interval greedoids. For details see Korte and Lovász [12].

2.8. *Ear-decomposition greedoids*

2.9. *Ear-decomposition greedoids of strong digraphs*

2.10. *Blossom greedoids*

2.11. *Paving greedoids*

3. Weak exchange properties for greedoids

In this section we formulate a weak version of the exchange property, which however together with (H1) and (H2) already defines greedoids.

(H3′) Whenever $A \subseteq E$, $x, y, z \in E - A$ such that $A \cup x \in \mathcal{F}$, $A \cup y \in \mathcal{F}$, $A \cup x \cup y \notin \mathcal{F}$ and $A \cup x \cup z \in \mathcal{F}$ then $A \cup y \cup z \in \mathcal{F}$.

Note first that (H3′) is an immediate consequence of (H3) by augmenting $A \cup y$ from $A \cup x \cup z$. The main result of this section is the converse of this observation:

Theorem 3.1. *Let* $(E, \mathcal{F})$ *be an accessible set-system satisfying* (H3′). *Then* $(E, \mathcal{F})$ *is a greedoid.*

Proof. Let $X, Y \in \mathcal{F}$, $|X| = |Y| + 1$. Let C be a largest subset of $X \cap Y$ in $\mathcal{F}$. We show (H3) by induction on $|X \cup Y| + |Y - C|$.

First we apply the induction hypothesis to Y and C, and obtain an $a \in Y - C$ such that $C \cup a \in \mathcal{F}$. Obviously $a \notin X$. Then apply the induction hypothesis repeatedly to augment $C \cup a$ from X to a set $Y' \subseteq X \cup a$ such that $C \cup a \subseteq Y'$, $|Y'| = |Y|$ and $Y' \in \mathcal{F}$.

Next we show that Y' can be augmented from X. This is trivial by induction hypothesis if $|X \cup Y'| < |X \cup Y|$, so assume that $|X \cup Y'| = |X \cup Y|$. It is also trivial if $X \cap Y'$ contains a larger set from $\mathcal{F}$ than C, so suppose this does not happen. Augment C from X by the induction hypothesis, to get an $x \in X - C$ such that $C \cup x \in \mathcal{F}$. Obviously, $x \notin Y'$. Then augment $C \cup x$ from Y' repeatedly, to get $X' \subseteq Y' \cup x$ such that $|X'| = |Y'|$, $C \cup x \subseteq X'$ and $X' \in \mathcal{F}$.

Case 1. $a \notin X'$. Then clearly $X' = Y' \cup x - a$. Let $A = X \cap Y' = Y' - a$, $y = a$ and $\{z\} = X - X'$. Then $A \cup x = X' \in \mathcal{F}$, $A \cup y = Y' \in \mathcal{F}$, $A \cup x \cup z = X \in \mathcal{F}$, so by (H3′), either $A \cup y \cup x$ or $A \cup y \cup z \in \mathcal{F}$. But this shows that Y' can be augmented from X.

Case 2. $a \in X'$. Since $C \cup x \subseteq X' \cap X$, we can use the induction hypothesis to augment X' from X by $u \in X - X'$. Then $C \cup a \subseteq Y' \cap (X' \cup u)$, so by the induction hypothesis, we can augment Y' from $(X' \cup u) - Y' \subseteq X - Y'$.

So we can augment Y' by an element $b \in X - Y'$. Then $C \cup a \subseteq Y \cap (Y' \cup b)$ and so by the induction hypothesis, we can augment Y from $Y' \cup b - Y \subseteq X - Y$. □

Remark. One may expect that (H3′) can be further weakened to the following:

(H3″) Whenever $A, A \cup x, A \cup y \in \mathcal{F}$, $A \cup x \cup y \notin \mathcal{F}$ and $A \cup x \cup z \in \mathcal{F}$, then $A \cup y \cup z \in \mathcal{F}$.

This is however not the case. Let $E = \{a, x, y, z\}$ and $\mathcal{F} = \{\emptyset, \{x\}, \{y\}, \{a, x\}, \{a, y\}, \{a, x, z\}\}$. Then (H3″) is satisfied but (H3′) fails for

$A = \{a\}$. It may be interesting to understand what kind of structure (H3″) imposes on an accessible set-system.

4. Transposition greedoids

Property (H3′) in the last section suggests a somewhat stronger property of accessible set-systems, which we call the *transposition property*.

(TP) Whenever $A \subseteq E$, $x, y \in E - A$ and $B \subseteq E - A - x - y$ such that $A \cup x \in \mathscr{F}$, $A \cup y \in \mathscr{F}$, $A \cup x \cup y \notin \mathscr{F}$ and $A \cup x \cup B \in \mathscr{F}$, then $A \cup y \cup B \in \mathscr{F}$.

Since (TP) is trivially stronger than (H3′), every accessible set-system with property (TP) is a greedoid. Such greedoids will be called *transposition greedoids*.

Similarly as in the previous section, we can formulate a weaker version of (TP):

(TP′) Whenever $A \subseteq E$, $x, y \in E - A$ and $B \subseteq E - A - x - y$ such that $A \in \mathscr{F}$, $A \cup x \in \mathscr{F}$, $A \cup y \in \mathscr{F}$, $A \cup x \cup y \notin \mathscr{F}$ and $A \cup x \cup B \in \mathscr{F}$, then $A \cup y \cup B \in \mathscr{F}$.

In contrast to the remark concluding that section, however, these properties are equivalent:

Theorem 4.1. *For every accessible set-system* $(E, \mathscr{F})$, *properties* (TP) *and* (TP′) *are equivalent.*

Proof. Obviously (TP) implies (TP′). to see the converse we use induction on $|E|$. So we may assume that the restriction of $\mathscr{F}$ to any proper subset of E satisfies (TP) and hence is a greedoid. Let $A \subseteq E$, $x, y \in E - A$ and $B \subseteq E - A - x - y$ such that $A \cup x \in \mathscr{F}$, $A \cup y \in \mathscr{F}$, $A \cup x \cup y \notin \mathscr{F}$ and $A \cup x \cup B \in \mathscr{F}$. Suppose, by way of contradiction, that $A \cup y \cup B \notin \mathscr{F}$. Set $A' = A \cup x \cup y$.

So we have a set $A' \notin \mathscr{F}$, two elements $x, y \in A'$ such that $A' - x \in \mathscr{F}$, $A' - y \in \mathscr{F}$ and $(A' - y) \cup B \in \mathscr{F}$ but $(A' - x) \cup B \notin \mathscr{F}$. Choose x and y with these properties so that $A' - x - y$ contains a subset $C \in \mathscr{F}$ with $|C|$ maximum.

Augment C from $A' - x$. By the maximality of C, we must augment by y, and hence $C \cup y \in \mathscr{F}$. Consider a maximal set C' such that $C \subset C' \subseteq A' - y$, $C' \in \mathscr{F}$ and $C' \cup y \in \mathscr{F}$. Note that $C' \neq A' - y$ since $A' \notin \mathscr{F}$. Augment C' from $A' - y$ by an element u. Then we have $C' \in \mathscr{F}$, $C' \cup u \in \mathscr{F}$, $C' \cup y \in \mathscr{F}$ but $C' \cup u \cup y \notin \mathscr{F}$ by the maximality of C'. Let $B' = A' - y - u - C'$. Then $C' \cup u \cup B' = A' - y \in \mathscr{F}$ and hence by (TP′), we have $C' \cup y \cup B' = A' - u \in \mathscr{F}$. Also, $C' \cup u \cup (B' \cup B) = (A' - y) \cup B \in \mathscr{F}$ and hence again by (TP′), $C' \cup y \cup (B' \cup B) = (A' - u) \cup B \in \mathscr{F}$. By replacing y by u we get a contradiction with the maximality of C, since $C \cup y \subseteq A' - x - u$. $\square$

Now we are going to reformulate the transposition property in the language framework. Let $(E, \mathscr{L})$ be a simple hereditary language. We introduce the following very special consequence of the exchange property:

(G4) If $\alpha xy\beta \in \mathscr{L}$ and $\alpha y \in \mathscr{L}$, then $\alpha yx\beta \in \mathscr{L}$.

It is easy to see that every hereditary language which is induced by an accessible set-system has property (G4). However, (G4) is not sufficient to characterize such hereditary languages.

Lemma 4.2. *A hereditary language is induced by some accessible set-system iff the following special case of the greedoid exchange property holds*:

(G5) *if $\alpha, \beta \in \mathscr{L}$, $\tilde{\alpha} \supset \tilde{\beta}$, then there exists an $x \in \tilde{\alpha}$ such that $\beta x \in \mathscr{L}$.*

Proof. The 'only if' part is trivial. Suppose that $(E, \mathscr{L})$ is a hereditary language with property (G5). Let

$$\mathscr{F} = \{\tilde{\alpha}: \alpha \in \mathscr{L}\}$$

and let

$$\mathscr{L}' = \{x_1 \cdots x_k: \{x_1, \ldots, x_i\} \in \mathscr{F} \text{ for } 1 \leqslant i \leqslant k\}.$$

We claim that $\mathscr{L} = \mathscr{L}'$. Clearly $\mathscr{L} \subseteq \mathscr{L}'$. To show equality, consider any $x_1 \cdots x_k \in \mathscr{L}'$, and let i be the largest index with $x_1 \cdots x_i \in \mathscr{L}$. We want to prove that $i = k$. Suppose not. Then $x_1 \cdots x_{i+1} \in \mathscr{L}'$ and so $\{x_1, \ldots, x_{i+1}\} \in \mathscr{F}$ and hence $\{1, \ldots, i+1\}$ has a permutation $(j_1, \ldots, j_{i+1})$ such that $x_{j_1} \cdots x_{j_{i+1}} \in \mathscr{L}$. Since $\{x_1, \ldots, x_i\} \subset \{x_{j_1}, \ldots, x_{j_{i+1}}\}$, we can augument $x_1 \cdots x_i$ from $x_{j_1} \cdots x_{j_{i+1}}$ to get a word in $\mathscr{L}$ by (G5). But the only element of $x_{j_1} \cdots x_{j_{i+1}}$ not in $x_1 \cdots x_i$ is x_{i+1}, so $x_1 \cdots x_{i+1} \in \mathscr{L}$, contradiction. $\square$

The *ordered version* of the *transposition property* consist of two parts:

(TP1) Let $\alpha x \in \mathscr{L}$ and $\alpha y \in \mathscr{L}$ but $\alpha xy \notin \mathscr{L}$, and let $\beta \in E^*$ such that $y \notin \beta$ and $\alpha x\beta \in \mathscr{L}$. Then $\alpha y\beta \in \mathscr{L}$.

(TP2) Let $\alpha x \in \mathscr{L}$, and $\alpha y \in \mathscr{L}$ but $\alpha xy \notin \mathscr{L}$, and let $\beta, \gamma \in E^*$ such that $\alpha x\beta y\gamma \in \mathscr{L}$. Then $\alpha y\beta x\gamma \in \mathscr{L}$.

If $(E, \mathscr{L})$ is induced by an accessible set-system, i.e., if (G5) holds, then (TP1) implies (TP2). In fact, $\alpha y\beta \in \mathscr{L}$ follows by (TP1) and $\alpha y\beta x\gamma \in \mathscr{L}$ follows by augmenting $\alpha y\beta$ from $\alpha x\beta y\gamma$.

For any simple hereditary language, (TP1) and (TP2) can jointly be formulated as follows:

(TP1 + 2) If $\alpha x, \alpha y \in \mathscr{L}$ but $\alpha xy \notin \mathscr{L}$, then the hereditary language $\mathscr{L}_x = \{\beta: \alpha x\beta \in \mathscr{L}\}$ is isomorphic to the hereditary language $\mathscr{L}_y = \{\beta: \alpha y\beta \in \mathscr{L}\}$, and in fact the mapping $\varphi: E - \tilde{\alpha} - x \to E - \tilde{\alpha} - y$ defined by

$$\varphi(u) = \begin{cases} x & \text{if } u = y, \\ u & \text{if } u \neq y \end{cases}$$

provides an isomorphism.

Theorem 4.3. *Let* $(E, \mathscr{L})$ *be a simple hereditary language having properties* (G4), (TP1) *and* (TP2). *Then* $(E, \mathscr{L})$ *is a transposition greedoid.*

Proof. It suffices to show that $(E, \mathscr{L})$ satisfies (G5); for then (TP1) implies that the corresponding accessible set-system satisfies (TP′) and hence is a transposition greedoid.

Claim. *Let* $\alpha\beta x\gamma \in \mathscr{L}$ *and* $\alpha x \in \mathscr{L}$. *Then there exist a word* β' *such that* $\tilde{\beta}' = \tilde{\beta}$ *and* $\alpha x\beta'\gamma \in \mathscr{L}$.

To prove the claim, consider a splitting $\beta = \beta_1\beta_2$ such that $\alpha\beta_1 x\beta_2'\gamma \in \mathscr{L}$ for some word β_2' with $\tilde{\beta}_2' = \tilde{\beta}_2$. Choose such a splitting with $|\beta_1|$ minimal. We want to show that $|\beta_1| = 0$. Suppose not, then write $\beta_1 = \beta_3 y$.

Case 1. $\alpha\beta_3 x \in \mathscr{L}$. Then, since $\alpha\beta_3 yx\beta_2'\gamma \in \mathscr{L}$, we have by (G4) that $\alpha\beta_3 xy\beta_2'\gamma \in \mathscr{L}$, which contradicts the minimality of β_1.

Case 2. $\alpha\beta_3 x \notin \mathscr{L}$. Let δ be the smallest beginning section of β_3 such that $\alpha\delta x \notin \mathscr{L}$, and with $\beta_3 = \delta\beta_4$. Clearly $|\delta| > 0$, so we can write $\delta = \delta_1 z$. Then $\alpha\delta_1 x \in \mathscr{L}$, $\alpha\delta_1 z \in \mathscr{L}$ but $\alpha\delta_1 zx \notin \mathscr{L}$. Furthermore, $\alpha\delta_1 z\beta_4 yx\beta_2'\gamma \in \mathscr{L}$, and hence by (TP2), we have $\alpha\delta_1 x\beta_4 yz\beta_2'\gamma \in \mathscr{L}$. But this contradicts the minimality of β_1 again. This completes the proof of the Claim.

Now we prove (G5). Let $\alpha, \beta \in \mathscr{L}$, $\tilde{\alpha} \supset \tilde{\beta}$. Let γ be the largest common beginning section of α and β. We prove by induction on $|\beta| - |\gamma|$ that there exists a letter $x \in \alpha$ such that $\beta x \in \mathscr{L}$. This is trivial if $\beta = \gamma$, so suppose that $\beta \neq \gamma$. Then we can write $\beta = \gamma y\beta_1'$ and $\alpha = \gamma\alpha_1 y\alpha_2$ since $\tilde{\beta} \subseteq \tilde{\alpha}$. By the Claim, there exists a word α_1' with $\tilde{\alpha}_1' = \tilde{\alpha}_1$ such that $\gamma y\alpha_1'\alpha_2 \in \mathscr{L}$. So by the induction hypothesis, there exists an $x \in (\gamma y\alpha_1'\alpha_2) = \tilde{\alpha}$ such that $\beta x \in \mathscr{L}$. $\square$

5. Examples of transposition greedoids

In this section we show that some of the non-interval greedoids listed in Section 2 are transposition greedoids while the others are not. First of all we remark

Theorem 5.1. *Interval greedoids have the transposition property.*

Proof. To verify (TP′), augument $A \cup y$ from $A \cup x \cup B$ to get $A \cup y \cup B'$ with $|B'| = |B|$ and $B' \subseteq B \cup x$. But $x \notin B'$ by the interval property. So $B' = B$. $\square$

Next we study perfect elimination greedoids. We will show that the language $(E, \mathscr{L})$ induced by elimination sequences, as defined in Section 2, satisfies (G4), (TP1) and (TP2). This will also give a shorter proof of the fact that this language is a greedoid. We need some simple lemmas.

Lemma 5.2. *Let xy and xy' be bisimplicial edges of a bipartite graph G. Then the transposition of $y \leftrightarrow y'$ is an automorphism of G.*

Proof. Trivial. □

Lemma 5.3. *Let $(x_1x_1', \ldots, x_kx_k')$ and $(x_1x_1'', \ldots, x_kx_k'')$ be two elimination sequences for the same word $x_1 \cdots x_k \in \mathcal{L}$. Then $G - \{x_1, \ldots, x_k, x_1', \ldots, x_k'\}$ is isomorphic to $G - \{x_1, \ldots, x_k, x_1'', \ldots, x_k''\}$ and there exists an isomorphism which keeps each point in $E - \{x_1, \ldots, x_k\}$ fixed.*

Proof. By induction on k. By Lemma 5.2 $x_1' \leftrightarrow x_1''$ is an automorphism of G, which, when restricted to $G - x_1 - x_1'$ gives the isomorphism with $G - x_1 - x_1''$. This isomorphism maps the elimination sequence $(x_2x_2', \ldots, x_kx_k')$ of $G - x_1 - x_1'$ onto an elimination sequence $(x_2x_2''', \ldots, x_kx_k''')$ of $G - x_1 - x_1''$. So by the induction hypothesis, $G - \{x_1, \ldots, x_k\} - \{x_1'', x_2''', \ldots, x_k'''\}$ has an isomorphism onto $G - \{x_1, \ldots, x_k\} - \{x_1'', \ldots, x_k''\}$ keeping $E - \{x_1, \ldots, x_k\}$ fixed. Combining this with the first isomorphism we obtain an isomorphism of $G - \{x_1, \ldots, x_k\} - \{x_1', \ldots, x_k'\}$ onto $G - \{x_1, \ldots, x_k\} - \{x_1'', \ldots, x_k''\}$ keeping $E - \{x_1, \ldots, x_k\}$ fixed. □

Lemma 5.4. *Any elimination sequence for $\alpha \in \mathcal{L}$ can be completed to an elimination sequence for $\alpha\beta \in \mathcal{L}$.*

Proof. Let $(a_1a_1', \ldots, a_ka_k')$ be an elimination sequence for α and $(a_1a_1'', \ldots, a_ka_k'', b_1b_1'', \ldots, b_kb_k'')$ be an elimination sequence for $\alpha\beta$. By Lemma 5.3, $G'' = G - \{a_1, \ldots, a_i, a_i'', \ldots, a_k''\}$ has an isomorphism onto $G' = G - \{a_1, \ldots, a_k, a_1', \ldots, a_k'\}$ keeping $E - \{a_1, \ldots, a_k\}$ fixed. This maps $(b_1b_1'', \ldots, b_mb_m'')$ onto an elimination sequence of G'. □

Theorem 5.5. *The hereditary language $(E, \mathcal{L})$ induced by elimination sequence satisfies* (G4), (TP1) *and* (TP2), *and hence it is a transposition greedoid.*

Proof. First we verify (G4). Let $\alpha xy\beta \in \mathcal{L}$ and $\alpha y \in \mathcal{L}$. Let $(a_1a_1', \ldots, a_ka_k')$ be an elimination sequence for α. By Lemma 5.4, this can be extended to an elimination sequence $(a_1a_1', \ldots, a_ka_k', xx', yy', b_1b_1', \ldots, b_mb_m')$ of $\alpha xy\beta$ and also to an elimination sequence $(a_1a_1', \ldots, a_ka_k', yy')$ of αy.

Case 1. $x' \neq y''$. Then $(a_1a_1', \ldots, a_ka_k', xx', yy'')$ is an elimination sequence for αxy, which can be extended to an elimination sequence $(a_1a_q', \ldots, a_ka_k', xx', yy'', b_1b_1'', \ldots, b_mb_m'')$ of $\alpha xy\beta$. But then $(a_1a_1', \ldots, a_ka_k', yy'', xx', b_1b_1'', \ldots, b_mb_m'')$ is an elimination sequence for $\alpha xy\beta$.

Case 2. $x' = y''$. Then by Lemma 5.2, the transposition $x \leftrightarrow y$ is an isomorphism of $G - \{a_1, \ldots, a_k, a_1', \ldots, a_k'\}$ and hence it maps the above elimination sequence for $\alpha xy\beta$ onto an elimination sequence for $\alpha yx\beta$.

We can verify (TP1) and (TP2) simultaneously. Let $\alpha x, \alpha y \in \mathscr{L}$ but $\alpha xy \notin \mathscr{L}$. Consider an elimination sequence $(a_1a_1', \ldots, a_ka_k')$ for α and extend it to an elimination sequence $(a_1a_1', \ldots, a_ka_k', xx')$ for αx and $(a_1a_1', \ldots, a_ka_k', yy'')$ for αy. Then $x' \neq y''$, since otherwise $(a_1a_1', \ldots, a_ka_k', xx', yy'')$ is an elimination sequence for αxy. But then by Lemma 5.2, the transposition $x \leftrightarrow y$ is an automorphism of $G - \{a_1, \ldots, a_k, a_1', \ldots, a_k'\}$, which means that it maps feasible continuations of α onto feasible words. $\square$

Series–parallel reduction greedoids also enjoy the transposition property. Again the fact that these languages are greedoids will also follow from the arguments below.

Theorem 5.6. *The hereditary language $(E, \mathscr{L})$ formed by series–parallel reduction sequences of a graph G satisfies* (G4), (TP1) *and* (TP2). *Consequently, it is a transposition greedoid.*

Proof. First we verify (G4). Without loss of generality we may assume that $|\alpha| = 0$. Suppose that $xy \in \mathscr{L}$ and $y \in \mathscr{L}$. We are going to show that $yx \in \mathscr{L}$ and $G \div x \div y \simeq G \div y \div x$. This will imply that if $xy\beta \in \mathscr{L}$ for some word β then also $yx\beta \in \mathscr{L}$. Unfortunately there are a number of cases to consider. Call two edges of G *adjoint to each other* if they are either in series or parallel.

Case 1. Suppose that both x and y have adjoints u *and* v and x, y, u, v are all distinct. Then v is in series (parallel) to y in $G \div x$ if it was in series (parallel) to y in G. Similar holds for x and u. Hence x is reducible in $G \div y$ and $G \div x \div y \simeq G \div y \div x$.

Case 2. Suppose that x and y have adjoints u, v but x, y, u, v are not all distinct. This results in 10 subcases as shown in Fig. 6. The lower rows show the graphs $G \div x \div y = G \div y \div x$.

Case 3. If one of x and y is a pending edge then the argument is trivial, since its contraction does not influence the reducibility of any other edge.

So we have verified (G4).

To show (TP1) and (TP2) we may again assume that $|\alpha| = 0$. Then from the hypothesis that $x \in \mathscr{L}$, $y \in \mathscr{L}$ but $xy \notin \mathscr{L}$, an easy argument shows that x and y have to be adjoint. Hence $G \div x \simeq G \div y$, and in fact there is an isomorphism which assigns x to y and any other edge to itself. Thus (TP1 + 2) is satisfied, and hence (TP1) and (TP2). $\square$

In Korte and Lovász [8] we showed by a rather lengthy argument that for any directed graph G, the language $(E, \mathscr{L})$ of retract sequences forms a greedoid.

Hence we will show the following more general result.

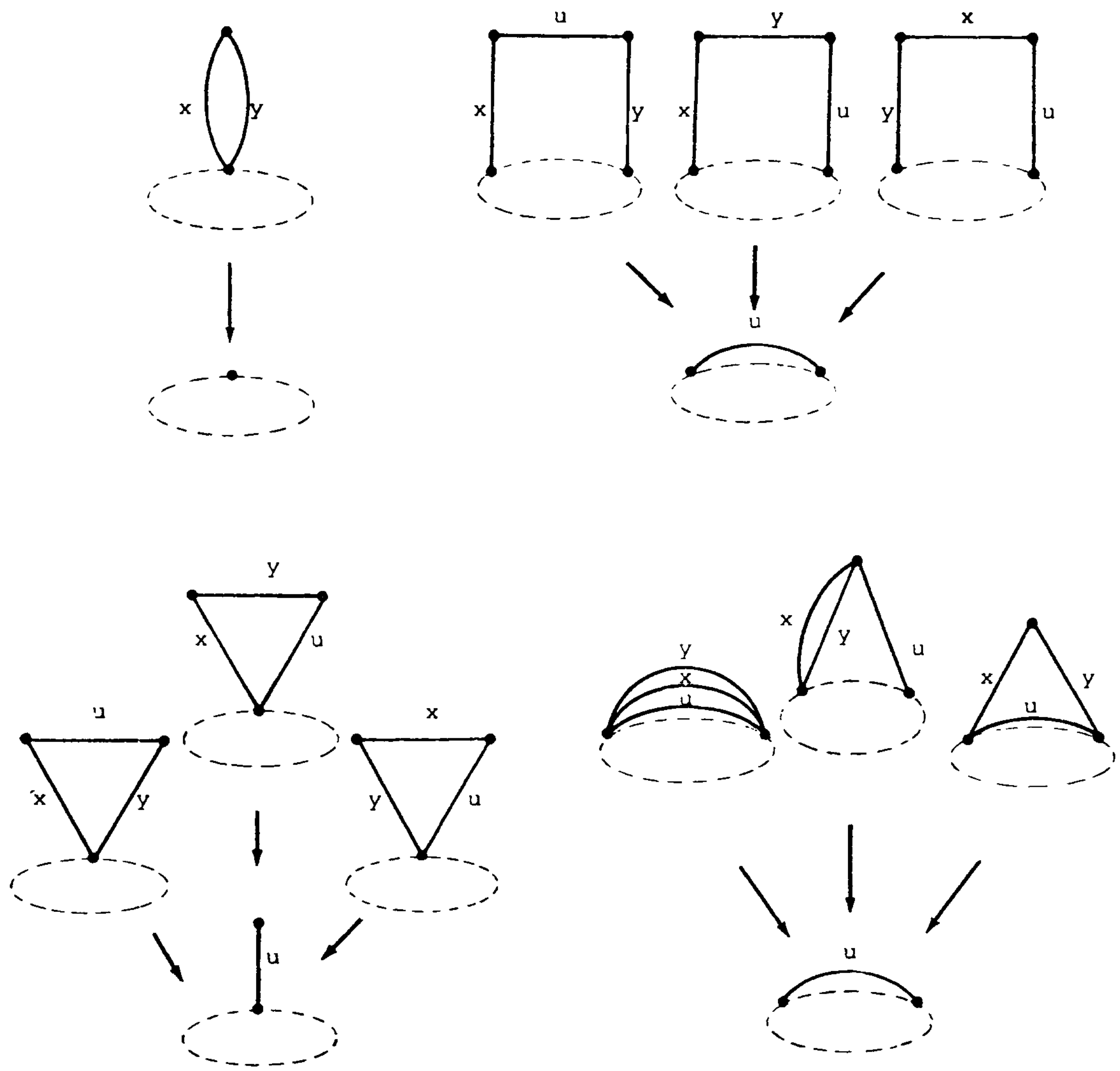

Fig. 6.

Theorem 5.7. *For any transformation monoid S, the accessible kernel* $(E, \mathcal{F})$ *of the set-system* $\{E - \varphi(E)\colon \varphi \in S, \varphi^2 = \varphi\}$ *satisfies* (TP) *and hence it forms a transposition greedoid.*

Proof. Let $A \cup x$, $A \cup y \in \mathcal{F}$, $A \cup x \cup y \notin \mathcal{F}$ and $A \cup x \cup B \in \mathcal{F}$. Let $A \cup x = E - \eta(E)$, $A \cup y = E - \mu(E)$ and $A \cup x \cup B = E - \varphi(E)$, where η, μ and φ are idempotent. We claim that $\eta(x) = y$ and $\mu(y) = x$. Suppose not, and let, say, $\mu(y) \neq x$. Then consider $\rho = (\eta \circ \mu)^k$ for a k such that ρ is idempotent (it is an elementary result in semigroup theory that such a k exists). Then $\rho(E) = E - A - x - y$. In fact, both η and μ keep $E - A - x - y$ fixed, and hence so does ρ. Hence $\rho(E) \supseteq E - A - x - y$. Furthermore, $\rho(E) = \mu(\eta((\eta \circ \mu)^{k-1}(E))) \subseteq \mu(E) = E - A - y$. So it remains to see that $x \notin \rho(E)$.

Suppose that $x \in \rho(E)$, then $x = \mu(z)$ for some $z \in \eta((\eta \circ \mu)^{k-1}(E)) \subseteq \eta(E) = E - A - x$. But if $z \in E - A - x - y$ then $\mu(z) = z \neq x$, and if $z = y$ then $\mu(z) \neq x$ by hypothesis. This contradiction proves that $\rho(E) = E - A - x - y$. However, this imples that $A \cup x \cup y \in \mathcal{F}$, contrary to the assumption. So we have proved that $\eta(x) = y$ and $\mu(y) = x$. Next we show that $\mu \circ \varphi \circ \eta$ is idempotent and $(\mu \circ \varphi \circ \eta)(E) = E - A - y - B$. Hence (TP) will follow immediately.

All three mappings η, φ and μ are identical on $E - A - x - y - B$ hence so is $\eta \circ \varphi \circ \mu$. Moreover,

$$(\mu \circ \varphi \circ \eta)(x) = (\mu \circ \varphi)(y) = \mu(y) = x.$$

So $\mu \circ \varphi \circ \eta$ is identical on $E - A - y - B$. Furthermore

$$(\mu \circ \varphi \circ \eta)(E) \subseteq (\mu \circ \varphi)(E) = \mu(E - A - x - B) = E - A - y - B.$$

Hence $\mu \circ \varphi \circ \eta$ is indeed idempotent and hence $A \cup y \cup B \in \mathcal{F}$. So (TP) holds. □

As a last example of transposition greedoids, we discuss dismantling greedoids.

Theorem 5.8. *The hereditary language $(E, \mathcal{L})$ formed by dismantling sequences of a digraph G satisfies* (G4) *and* (TP1 + 2). *Hence $(E, \mathcal{L})$ is a transposition greedoid.*

Proof. Let $\alpha xy\beta \in \mathcal{L}$ and $\alpha y \in \mathcal{L}$. Without loss of generality, $|\alpha| = |\beta| = 0$. Since $xy \in \mathcal{L}$ there is a point u adjacent to x such that x is retractable to u in G and a point $v \neq x$ adjacent to y such that y is retractable to v in $G - x$. Furthermore, there exists a point v' such that y is adjacent to v' and y is retractable to v' in G. If $u \neq y$ then x is still retractable to u in $G - y$ and hence $yx \in \mathcal{L}$. So suppose that $y = u$. So x and y are adjacent, say $xy \in E(G)$. If $v' \neq x$ then since y is retractable to v', we have $xv' \in E(G)$, and clearly z is retractable to v' in $G - y$. So $yx \in \mathcal{L}$. Finally, if $v' = x$ then v is adjacent to x since v is adjacent to y and xy is retractable to x. Since clearly x is retractable to v in $G - y$, we obtain again that $yx \in \mathcal{L}$. This proves (G4).

To prove (TP1 + 2), we may assume again that $|\alpha| = 0$. The same argument as above implies that if $x \in \mathcal{L}$, $y \in \mathcal{L}$ but $xy \notin \mathcal{L}$ then x and y are adjacent and mutually retractable to each other. Hence $G - x \simeq G - y$ and an isomorphism is given by

$$\varphi(u) = \begin{cases} x & \text{if } u = y, \\ u & \text{if } u \neq y. \end{cases}$$

This proves (TP1 + 2). □

Remark. It appears from the above discussion that dismantling greedoids are strongly related to retract greedoids. The construction of dismantling greedoids can be generalized to transformation monoids as follows. Let S be a transformation monoid on a set E and let R be a binary relation on E invariant under S. Call a sequence $(x_1, \ldots, x_k)$ of points a *dismantling sequence,* if there exist idempotents $\varphi_1, \ldots, \varphi_k$ of S such that $E - \varphi_i(E) = \{x_1, \ldots, x_i\}$ for $i = 1, \ldots, k-1$ and moreover $(x_i, \varphi(x_i)) \in R$ for each i.

Then the dismantling sequences form a transposition greedoid. This can be proved in the same way as above.

Let $(E, \mathcal{F})$ be any greedoid. We define the *completion* of $(E, \mathcal{F})$ as the pair $(E, \bar{\mathcal{F}})$ where $\bar{\mathcal{F}} = \mathcal{F} \cup \{X \subseteq E \mid X \text{ is spanning}\}$. It is easy to see that $(E, \bar{\mathcal{F}})$ is a greedoid. In general, completion does not preserve the interval property. For example the completion of the matroid on $E = \{x, y, z\}$ with two bases $\{x, y\}$ and $\{x, z\}$ does not have the interval property. However completion does preserve the transposition property.

Theorem 5.9. *If $(E, \mathcal{F})$ has the transposition property then its completion $(E, \bar{\mathcal{F}})$ also has it.*

Proof. Suppose that $A \cup x$, $A \cup y \in \bar{\mathcal{F}}$, $A \cup x \cup y \notin \bar{\mathcal{F}}$ and $A \cup x \cup B \in \bar{\mathcal{F}}$. Clearly $A \cup x$ cannot be spanning, since the $A \cup x \cup y$ would also be spanning. So $A \cup x \in \mathcal{F}$ and similarily $A \cup y \in \mathcal{F}$. If $A \cup x \cup B \in \mathcal{F}$, then $A \cup y \cup B \in \mathcal{F}$ by the transposition property of $(E, \mathcal{F})$. If $A \cup x \cup B$ is spanning, then augument $A \cup x$ to a basis $A \cup x \cup W$ with $W \subseteq B$. Then by the transposition property of $(E, \mathcal{F})$, $A \cup y \cup W \in \mathcal{F}$, so $A \cup y \cup W$ is a basis and hence $A \cup y \cup B$ is spanning. □

Using this theorem, we can construct further transposition greedoids, like completions of matroids.

6. Some examples of non-transposition greedoids

The bipartite matching greedoid of the graph in Fig. 7 does not have the transposition property. Here $\{x\}, \{y\} \in \mathcal{F}$, $\{x, y\} \notin \mathcal{F}$, $\{x, b, c\} \in \mathcal{F}$ but $\{y, b, c\} \notin \mathcal{F}$. As mentioned above, bipartite matching greedoids are special cases of Gaussian elimination greedoids. Therefore the latter do not enjoy the transposition property either.

It is interesting to contrast this with Theorem 5.5, which states that the closely related perfect elimination greedoids all have the transposition property.

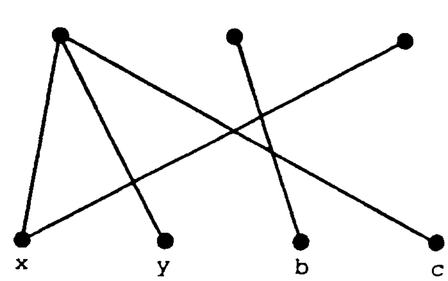

Fig. 7.

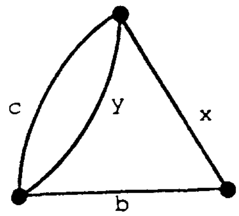

Fig. 8.

Next consider the circuit matroid of the graph in Fig. 8, twisted by $A = \{b\}$. Then again x, y and $\{x, b, c\}$ are feasible but $\{x, y\}$ *and* $\{y, b, c\}$ are not, showing that twisted matroids are not transposition greedoids.

Similar counterexamples can be constructed for all other of non-interval greedoids of Section 2 which were not proven to be transposition greedoids.

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